Sylvester Waves in the Coxeter Groups

Leonid G. Fel[†] and Boris Y. Rubinstein[‡]

[†]School of Physics and Astronomy,
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
e-mail: lfel@ccsg.tau.ac.il
and
[‡]Department of Engineering Sciences and Applied Mathematics
Northwestern University, 2145 Sheridan Road
Evanston IL 60208-3125, U.S.A.

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Abstract

A new recursive procedure of the calculation of partition numbers function $W(s,\mathbf{d}^m)$ is suggested. We find its zeroes and prove a lemma on the function parity properties. The explicit formulas of $W(s,\mathbf{d}^m)$ and their periods $\tau(G)$ for the irreducible Coxeter groups and a list for the first ten symmetric group $\mathcal{S}_m$ are presented. A *least common multiple* $\mathcal{L}(m)$ of the series of the natural numbers 1,2,..,m plays a role of the period $\tau(\mathcal{S}_m)$ of $W(s,\mathbf{d}^m)$ in $\mathcal{S}_m$. An asymptotic behaviour of $\mathcal{L}(m)$ with $m\to\infty$ is found.

Pacs: Number theory, Invariant theory

1 Introduction

More than hundred years ago J.J.Sylvester stated [10, 11] and proved [12] a theorem about restricted partition number $W(s, \mathbf{d}^m)$ of positive integer s with respect to the m-tuple of positive integers $\mathbf{d}^m = \{d_1, d_2, ..., d_m\}$:

Theorem. *The number $W(s, \mathbf{d}^m)$ of ways in which s can be composed of (not necessarily distinct) m integers $d_1, d_2, ..., d_m$ is made up of a finite number of waves*

$$W(s, \mathbf{d}^m) = \sum_{q}^{max\ q} W_q(s, \mathbf{d}^m) \ , \quad W_q(s, \mathbf{d}^m) = \sum_{k}^{max\ k} W_{p_k|q}(s, \mathbf{d}^m) \ , \tag{1}$$

where q run over all distinct factors in $d_1, d_2, ..., d_m$ and $W_{p_k|q}(s, \mathbf{d}^m)$ denotes the coefficient of t^{-1} in the series expansion in ascending powers of t of

$$F(s, \mathbf{d}^m, k; t) = \ e^{sw_k} \ \prod_{r=1}^{m} \frac{1}{1 - e^{d_r u_k}} \ , \quad w_k = 2\pi i \ \frac{p_k}{q} + t \ , \quad u_k = 2\pi i \ \frac{p_k}{q} - t \ , \tag{2}$$

and $p_1, p_2, ..., p_{max\ k}$ are all numbers (unity included) less than q and prime to it.

$W(s, \mathbf{d}^m)$ is also a number of sets of positive integer solutions $(x_1, x_2, ..., x_m)$ of equation $\sum_r^m d_r x_r = s$. It is known that $W(s, \mathbf{d}^m)$ is equal to the coefficient of t^s in the expansion of generating function

$$M(\mathbf{d}^m, t) = \prod_{r=1}^{m} \frac{1}{1 - t^{d_r}} = \sum_{s=0}^{\infty} W(s, \mathbf{d}^m)\ t^s \ . \tag{3}$$

If the exponents $d_1, d_2, ..., d_m$ become the series of integers $1, 2, 3, ..., m$, the number of waves is m and $W(s, \mathbf{d}^m)$ of s is usually referred to as a restricted partition number $\mathcal{P}_m(s)$ of s into parts none of which exceeds m.

Another definition of $W(s, \mathbf{d}^m)$ comes from the polynomial invariant of finite reflection groups. Let $M(\mathbf{d}^m, t)$ is a Molien function of such a group G, d_r are the degrees of basic invariants, and m is the number of basic invariants [8]. Then $W(s, \mathbf{d}^m)$ gives a number of algebraic independent polynomial invariants of the s-degree for group G.

Throughout his papers J.J.Sylvester gave different names for $W(s, \mathbf{d}^m)$: *quotity*, *denumerant*, *quot-undulant* and *quot-additant*. Sometime after he discarded some of them. Because of a wide usage of $W(s, \mathbf{d}^m)$ not only as a partition number we shall call $W(s, \mathbf{d}^m)$ a *Sylvester wave*.

The Sylvester theorem is a very powerful tool not only in the trivial situation when m is finite but also it was used for the purposes of asymptotic evaluations $\mathcal{P}_m(s)$, as well as for the main term of the Hardy-Ramanujan formulas for unrestricted partition number $\mathcal{P}(s)$ [13].

Recent progress in the self-dual problem of effective isotropic conductivity in two-dimensional three-component regular checkerboards [5] and its further extension on the m-component anisotropic cases [6] have shown an existence of algebraic equations with permutation invariance with respect to the action of the finite group G permuting m components. G is a subgroup of symmetric group $\mathcal{S}_m$ and the coefficients in the equations are build out of algebraic independent polynomial invariants for group G. Here $W(s, \mathbf{d}^m)$ measures a degree of non-universality of the algebraic solution with respect to the different kinds of m-color plane groups.

Several proofs of Sylvester theorem are known [12],[3]. All of them make use of the Cauchy's theory of residues. The recursion relations imposed on $W(s,\mathbf{d}^m)$ provide a combinatorial version of Sylvester formula. The classical example for the elementary (complex-variable-free) derivation was shown by Erdös [4] for the main term of the Hardy-Ramanujan formula. Recently an elementary derivation of Szekeres' formula for $W(s,\mathbf{d}^m)$ based on the recursion satisfied by $W(s,\mathbf{d}^m)$ was elaborated in [2]. In this paper we give a new derivation of the Sylvester waves based on the recursion relation for $W(s,\mathbf{d}^m)$. We find also its *zeroes* and prove a lemma on parity properties of the Sylvester waves. Finally we present a list of the first ten Sylvester waves $W(s,\mathcal{S}_m)$, $m=1,...,10$ for symmetric groups $\mathcal{S}_m$ and for all Coxeter groups. In the Appendix we prove a conjecture on asymptotic behaviour of the *least common multiple* $\mathsf{lcm}(1,2,...,N)$ of the series of natural numbers.

2 Recursion relation for $W(s,\mathbf{d}^m)$.

We start with a recursion that follows from (3)

$$M(\mathbf{d}^m,t)-M(\mathbf{d}^{m-1},t)=t^{d_m}M(\mathbf{d}^m,t), \tag{4}$$

and after inserting the series expansions into the last equation we arrive at

$$W(s,\mathbf{d}^m)=W(s,\mathbf{d}^{m-1})+W(s-d_m,\mathbf{d}^m)\ ,\ \ d_m\le s\ , \tag{5}$$

where s is assumed to be real. We apply now the recursive procedure (5) several times

$$W(s,\mathbf{d}^m)=\sum_{p=0}^{r_m}W(s-p\cdot d_m,\mathbf{d}^{m-1})+W(s-(r_m+1)\cdot d_m,\mathbf{d}^m)\ . \tag{6}$$

Let us consider *the generic form* of $W(k\cdot\tau\{\mathbf{d}^m\}+s,\mathbf{d}^m),\ s<\tau\{\mathbf{d}^m\}$ where $k,\ s$ and $\tau\{\mathbf{d}^m\}$ are the independent positive integers. We will choose them in such a way that

$$k\cdot\tau\{\mathbf{d}^m\}+s-(r_m+1)\cdot d_m=(k-1)\cdot\tau\{\mathbf{d}^m\}+s\ ,\quad\Rightarrow\quad\tau\{\mathbf{d}^m\}=(r_m+1)\cdot d_m\ . \tag{7}$$

Thus the relation (6) reads

$$\begin{aligned} W(k\cdot\tau\{\mathbf{d}^m\}+s,\mathbf{d}^m) &= W(\ (k-1)\cdot\tau\{\mathbf{d}^m\}+s,\mathbf{d}^m)+ \\ &\sum_{p=0}^{\delta_m-1}W(k\cdot\tau\{\mathbf{d}^m\}-p\cdot d_m+s,\mathbf{d}^{m-1})\ \ ,\quad\delta_m=\frac{\tau\{\mathbf{d}^m\}}{d_m}\ . \end{aligned} \tag{8}$$

As follows from (7), in order to return via the recursive procedure from $W(k\cdot\tau\{\mathbf{d}^m\}+s,\mathbf{d}^m)$ to $W(\ (k-1)\cdot\tau\{\mathbf{d}^m\}+s,\mathbf{d}^m)$ we must use $\tau\{\mathbf{d}^m\}$ which have d_m as a divisor. Due to the arbitrariness of d_m it is easy to conclude that all exponents $d_1,d_2,...,d_m$ serve as the divisors of $\tau\{\mathbf{d}^m\}$. In other words $\tau\{\mathbf{d}^m\}$ is the *least common multiple* lcm of the exponents $d_1,d_2,...,d_m$

$$\tau\{\mathbf{d}^m\}=\mathsf{lcm}(d_1,d_2,...,d_m)\ . \tag{9}$$

Actually $\tau\{\mathbf{d}^m\}$ does play a role of the "*period* " of $W(s,\mathbf{d}^m)$. But strictly speaking it is not a periodic function with respect to the integer variable s as could be seen from (8). The rest of the paper clarifies this hidden periodicity.

As we have mentioned above, $W(s,\mathbf{d}^m)$ gives a number of algebraic independent polynomial invariants of the s-degree for the group G. The situation becomes more transparent if we deal with the irreducible Coxeter group where the degrees d_r and the number of basic invariants m are well known.

Table 1. The "*periods* " $\tau(G)$ of $W(s,\mathbf{d}^m)$ for the irreducible Coxeter groups.

G	A_m	B_m	D_m	G_2	F_4	E_6
$\tau(G)$	$\mathcal{L}(m+1)$	$2\mathcal{L}(m)$	$2\mathcal{L}(m)$	6	24	360
G	E_7	E_8	H_3	H_4	I_2(2m)	I_2(2m+1)
$\tau(G)$	2520	2520	30	60	2 m	2 (2m+1)

where $\mathcal{L}(m)$=lcm(1,2,3,...,m) is the *least common multiple* of the series of the natural numbers.

$\mathcal{L}(m)$ can be viewed as $\tau(\mathcal{S}_m)$ for symmetric group $\mathcal{S}_m$ or, in other words, as a "*period* " of the restricted partition number $\mathcal{P}_m(s)$. This makes it possible to pose a question about asymptotic behaviour of $\tau(\mathcal{S}_m)$ with $m\to\infty$. $\mathcal{L}(m)$ is a very fast growing function: $\mathcal{L}(1)$=1, $\mathcal{L}(10)$=2520, $\mathcal{L}(20)$=232792560, $\mathcal{L}(30)$=2329089562800 etc. Actually $\frac{\ln\mathcal{L}(m)}{m}$ oscillates infinitely many times around 1 and the function $\mathcal{L}(m)$ has an exponential increase with the asymptotic law [1]

$$\lim_{m\to\infty}\frac{\ln\mathcal{L}(m)}{m}=1. \tag{10}$$

3 Polynomial representation for $W(s,\mathbf{d}^m)$.

Making use of the relations (8,9) we obtain the exact formula for $W(k\cdot\tau\{\mathbf{d}^m\}+s,\mathbf{d}^m)$ for different $\mathbf{d}^m$. We will treat it in an ascending order in the number m of exponents. The first steps are simple and they yield

$\underline{\mathbf{d}^1=(d_1)}$, $\tau\{\mathbf{d}^1\}>s\geq 0$

$$W(k\cdot d_1+s,\mathbf{d}^1)=W(s,\mathbf{d}^1)=\Psi_{d_1}(s)=\left\{\begin{array}{lll}1 \ , & s=0 & (\mathrm{mod}\ d_1)\\ 0 \ , & s\neq 0 & (\mathrm{mod}\ d_1)\end{array}\right. \tag{11}$$

$\Psi_{d_1}(s)$ may be represented as a sum of prime roots of unit of degree d_1:

$$\Psi_{d_1}(s)=\frac{1}{d_1}\sum_{k=0}^{d_1-1}\exp(\frac{2\pi iks}{d_1})=\frac{1}{d_1}\left\{\begin{array}{ll}1+\cos\pi s+2\sum_{k=1}^{d_1/2-1}\cos\frac{2\pi ks}{d_1}, & \text{even } d_1\\ 1+2\sum_{k=1}^{(d_1-1)/2}\cos\frac{2\pi ks}{d_1}, & \text{odd } d_1\end{array}\right. .$$

$\underline{\mathbf{d}^2=(d_1,d_2)}$, $\tau\{\mathbf{d}^2\}>s\geq 0$

$$W(k\cdot\tau\{\mathbf{d}^2\}+s,\mathbf{d}^2)=W(s,\mathbf{d}^2)+k\cdot\sum_{p=0}^{\delta_2-1}W(|s-p\ d_2|,\mathbf{d}^1)\ . \tag{12}$$

[1]It seems to be strange but we have not found throughout the textbooks on number theory any discussion about the asymptotics of lcm(1,2,3,...,m). The formula (10) was conjectured by one of the authors (LGF) based on the numerical calculations and proved by Z.Rudnick (Tel-Aviv Univ., Israel) which had communicated this proof to us. The proof is given in Appendix A.

$\underline{\mathbf{d}^3 = (d_1, d_2, d_3)}$, $\tau\{\mathbf{d}^3\} > s \geq 0$

$$W(k \cdot \tau\{\mathbf{d}^3\} + s, \mathbf{d}^3) = W(s, \mathbf{d}^3) + k \cdot \sum_{p=0}^{\delta_3 - 1} W(|s - p\ d_3|, \mathbf{d}^2) + \frac{k(k+1)}{2} \frac{\tau\{\mathbf{d}^3\}}{\tau\{\mathbf{d}^2\}} \sum_{p=0}^{\delta_3 - 1} \sum_{q=0}^{\delta_2 - 1} W(|s - p\ d_3 - q\ d_2|, \mathbf{d}^1)\ . \qquad (13)$$

Now it is simple to deduce by induction that in the general case $W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m)$ has a polynomial representation with respect to k

$$W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m) = A^m_{m-1}(s)\ k^{m-1} + A^m_{m-2}(s)\ k^{m-2} + ... + A^m_1(s)\ k + A^m_0(s, \mathbf{d}^m)\ , \quad (14)$$

where $A^m_{m-r}(s)$ is based on the $\tau\{\mathbf{d}^r\}$-periodic functions as well as the entire $W(s, \mathbf{d}^m)$ is based on the $\tau\{\mathbf{d}^m\}$-periodic functions. The coefficient in the leading term can be written in a closed form

$$A^m_{m-1}(s) = \frac{1}{(m-1)!} \cdot \frac{\tau^{m-2}\{\mathbf{d}^m\}}{\tau\{\mathbf{d}^2\} \cdot \tau\{\mathbf{d}^3\} \cdot ... \cdot \tau\{\mathbf{d}^{m-1}\}} \times$$
$$\times \sum_{p=0}^{\delta_m - 1} \sum_{q=0}^{\delta_{m-1} - 1} ... \sum_{v=0}^{\delta_2 - 1} W(|s - p\ d_m - q\ d_{m-1} - ... - v\ d_2|, \mathbf{d}^1)\ . \qquad (15)$$

With $d_1 = 1$ we have $W(|s - p\ d_m - q\ d_{m-1} - ... - v\ d_2|, 1) = 1$, which makes $A^m_{m-1}(s)$ independent of s and gives an asymptotics of $W(s, \mathbf{d}^m)$ for $s \gg m$

$$A^m_{m-1}(s) = \frac{\tau^{m-1}\{\mathbf{d}^m\}}{(m-1)!\ m!}\ , \quad W(s, \mathbf{d}^m) \overset{s \to \infty}{\simeq} \frac{s^{m-1}}{(m-1)!\ m!}\ . \qquad (16)$$

Now we are ready to prove the statement about splitting of $W(s, \mathbf{d}^m)$ into periodic and non-periodic parts.

Lemma 3.1. *The Sylvester wave $W(s, \mathbf{d}^m)$ can be represented in the following way*

$$W(s, \mathbf{d}^m) = Q^m_m(s) + \sum_{j=1}^{m-1} Q^m_j(s) \cdot s^{m-j}\ , \qquad (17)$$

where $Q^m_j(s)$ is a periodic function with the period $\tau\{\mathbf{d}^j\} = \mathsf{lcm}(d_1, d_2, ..., d_j)$.

Proof. We start with the identity for the polynomial representation for $W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m)$

$$W((k+1) \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m) = W(k \cdot \tau\{\mathbf{d}^m\} + s + \tau\{\mathbf{d}^m\}, \mathbf{d}^m)\ ,$$

that can be transformed, using (14), into

$$A^m_{m-1}(s)\ (k+1)^{m-1} + A^m_{m-2}(s)\ (k+1)^{m-2} + ... + A^m_1(s)\ (k+1) + W(s, \mathbf{d}^m) =$$
$$A^m_{m-1}(s + \tau\{\mathbf{d}^m\})\ k^{m-1} + A^m_{m-2}(s + \tau\{\mathbf{d}^m\})\ k^{m-2} + ... + A^m_1(s + \tau\{\mathbf{d}^m\})\ k + W(s + \tau\{\mathbf{d}^m\}, \mathbf{d}^m)\ . \quad (18)$$

The last identity generates a finite number of coupled difference equations for the coefficients $A_r^m(s)$

$$A_{m-r}^m(s+\tau\{\mathbf{d}^m\}) = \sum_{j=1}^{r} C_{m-j}^{m-r}\cdot A_{m-j}^m(s)\ , \quad 1\le r\le m\ , \tag{19}$$

where C_n^k denotes a binomial coefficient. The first equation ($r=1$)

$$A_{m-1}^m(s+\tau\{\mathbf{d}^m\}) = A_{m-1}^m(s)$$

declares that $A_{m-1}^m(s)$ is an arbitrary $\tau\{\mathbf{d}^m\}$-periodic function. We can specify the last statement taking into account (14) that actually $A_{m-1}^m(s)$ is $\tau\{\mathbf{d}^1\}$-periodic function which will be denoted as $Q_1^m(s)$. The second equation ($r=2$)

$$A_{m-2}^m(s+\tau\{\mathbf{d}^m\}) = A_{m-2}^m(s) + (m-1)\cdot A_{m-1}^m(s)$$

can be solved completely

$$A_{m-2}^m(s) = Q_2^m(s) + (m-1)\cdot s\cdot Q_1^m(s)\ , \tag{20}$$

where $Q_2^m(s+\tau\{\mathbf{d}^2\}) = Q_2^m(s)$. Continuing this procedure, it is not difficult to prove by induction that for any r we have

$$A_{m-r}^m(s) = \sum_{j=1}^{r} C_{m-j}^{m-r}\cdot Q_j^m(s)\cdot s^{r-j}\ , \tag{21}$$

where $Q_j^m(s+\tau\{\mathbf{d}^j\}) = Q_j^m(s)$. Since $W(s,\mathbf{d}^m) = A_0^m(s)$ we arrive finally at (17) by inserting $r=m$ into equation (21), that splits $W(s,\mathbf{d}^m)$, in accordance with the Sylvester theorem, into periodic and non-periodic parts. ■

4 Partition identities and zeroes of $W(s,\mathbf{d}^m)$.

In this section we assume that the variable s has only integer values.

Consider a new quantity

$$V(s,\mathbf{d}^m) = W(s-\xi\{\mathbf{d}^m\},\mathbf{d}^m)\ ,\ \ \xi\{\mathbf{d}^m\} = \frac{1}{2}\sum_{i=1}^{m} d_i\ . \tag{22}$$

Lemma 4.1. *$V(s,\mathbf{d}^m)$ has the following parity properties:*

$$V(s,\mathbf{d}^{2m}) = -V(-s,\mathbf{d}^{2m}),\quad V(s,\mathbf{d}^{2m+1}) = V(-s,\mathbf{d}^{2m+1}). \tag{23}$$

Proof. A basic recursion relation (5) can be rewritten for $V(s,\mathbf{d}^m)$

$$V(s,\mathbf{d}^m) - V(s-d_m,\mathbf{d}^m) = V(s-\frac{d_m}{2},\mathbf{d}^{m-1})\ . \tag{24}$$

The last relation produces two equations in a new variable $q = s-\frac{d_m}{2}$

$$V(q,\mathbf{d}^{m-1}) \ \ = \ \ V(q+\frac{d_m}{2},\mathbf{d}^m) - V(q-\frac{d_m}{2},\mathbf{d}^m),$$

$$V(-q,\mathbf{d}^{m-1}) \quad = \quad V(-q+\frac{d_m}{2},\mathbf{d}^m) - V(-q-\frac{d_m}{2},\mathbf{d}^m)\ . \tag{25}$$

Hence if $V(q,\mathbf{d}^m)$ is an even function of q, then $V(q,\mathbf{d}^{m-1})$ is an odd one, and vice versa. Because $V(q,\mathbf{d}^1)$ is an even function, we arrive at (23). ■

Corollary. *If* $s_1+s_2+2\xi\{\mathbf{d}^m\}=0$ *, then*

$$W(s_1,\mathbf{d}^m) = (-1)^{m+1}W(s_2,\mathbf{d}^m)$$

Proof. This follows from the parity properties and after substitution two new variables $s_1=s-\xi\{\mathbf{d}^m\}$, $s_2=-s-\xi\{\mathbf{d}^m\}$ into (23). ■

Lemma 4.2. *Let m-tuple $\{\mathbf{d}^m\}$ generates the Sylvester wave $W(s,\mathbf{d}^m)$. Then for every integer p a m-tuple $\{p\cdot\mathbf{d}^m\}=\{pd_1,pd_2,...,pd_m\}$ generates the following Sylvester wave*

$$W(s,p\cdot\mathbf{d}^m) = \Psi_p(s)\cdot W(\frac{s}{p},\mathbf{d}^m)\ , \quad \text{or} \quad V(s,p\cdot\mathbf{d}^m) = \Psi_p(s-p\xi\{\mathbf{d}^m\})\cdot V(\frac{s}{p},\mathbf{d}^m)\ , \tag{26}$$

where the periodic function $\Psi_p(s)=\Psi_p(s+p)$ is defined in (11).

Proof. According to the definition (3)

$$\sum_s W(s,p\cdot\mathbf{d}^m)\cdot t^s = \sum_s W(s,\mathbf{d}^m)\cdot t^{ps} = \sum_{s'} W(\frac{s'}{p},\mathbf{d}^m)\cdot t^{s'}$$

Equating powers of t in the latter equation and taking into account that s'/p must be integer we obtain (26). ■

Lemma 4.3. *Let m-tuple $\{\mathbf{d}^m\}$ generates the Sylvester wave $W(s,\mathbf{d}^m)$. Then $W(s,\mathbf{d}^m)$ has the following zeroes:*

- *If all exponents d_r are mutually prime numbers, then the zeroes $\mathfrak{s}_0(\mathbf{d}^m)$ read*

$$\begin{aligned}\mathfrak{s}_0(\mathbf{d}^m) \quad &= \quad -1,-2,...,-\sum_{r=1}^m d_r+1\ , \quad \text{if} \quad m=2k+1\ ,\\ \mathfrak{s}_0(\mathbf{d}^m) \quad &= \quad -1,-2,...,-\sum_{r=1}^m d_r+1\ ,\ -\xi\{\mathbf{d}^m\}\ , \quad \text{if} \quad m=2k\ ;\end{aligned} \tag{27}$$

- *If all exponents d_r have a maximal common factor p, then $W(s,\mathbf{d}^m)$ has infinite number of zeroes $\mathfrak{S}_1(\mathbf{d}^m)$ which are distributed in the following way*

$$\mathfrak{S}_1(\mathbf{d}^m) = \mathfrak{s}_1(\mathbf{d}^m)\cup\{\mathbb{Z}/p\mathbb{Z}\}\quad , \tag{28}$$

where $\{\mathbb{Z}/p\mathbb{Z}\}$ denotes a set of integers $\mathbb{Z}$ with deleted integers of modulo p

$$\{\mathbb{Z}/p\mathbb{Z}\} = \{...,-p-1,-p+1,...,-1,1,...,p-1,p+1,...\} \tag{29}$$

and

$$\mathfrak{s}_1(\mathbf{d}^m) \;=\; -p, -2p, ..., -\sum_{r=1}^{m} d_r + p\;, \quad \text{if} \quad m = 2k+1\;,$$
$$\mathfrak{s}_1(\mathbf{d}^m) \;=\; -p, -2p, ..., -\sum_{r=1}^{m} d_r + p,\; -\xi\{\mathbf{d}^m\}\;, \;\text{if} \quad m = 2k\;. \tag{30}$$

<u>*Proof.*</u> Consider again the relation (6) which we rewrite as follows

$$\sum_{s=0}^{\infty} W(s,\mathbf{d}^m)\cdot t^s = \frac{1}{1-t^{d_m}}\cdot\sum_{s'=0}^{\infty} W(s',\mathbf{d}^{m-1})\cdot t^{s'} \tag{31}$$

assuming that the exponents in $\mathbf{d}^m$ are sorted in the ascending order. Note that the influence of the new d_m exponent appears only in terms t^s with $s \geq d_m$. This enables us to deduce that the values of $W(s,\mathbf{d}^{m-1})$ and $W(s,\mathbf{d}^m)$ coincide at integer positive values $s = 0, 1, \ldots, d_m - 1$. This means that for $0 \leq s \leq d_m - 1$ we have $W(s,\mathbf{d}^m) = W(s,\mathbf{d}^{m-1})$. Recalling the main recursion relation (5) we conclude that

$$W(s,\mathbf{d}^m) = 0 \;\; (-d_m \leq s \leq -1).$$

Using the last relation for m and $m-1$ in (5) we can find also

$$W(s-d_m,\mathbf{d}^m) = 0 \;\; (-d_{m-1} \leq s \leq -1) \quad\Rightarrow\quad W(s,\mathbf{d}^m) = 0 \;\; (-d_{m-1} - d_m \leq s \leq -1).$$

Repeating this procedure and taking into account that at the last step it leads to the zeroes of Ψ_{d_1} which are located at $(1 - d_1 \leq s \leq -1)$, we get the set of the zeroes for $W(s,\mathbf{d}^m)$ with odd number of exponents $m = 2k+1$

$$W(s,\mathbf{d}^m) = 0 \;\; (1 - \sum_{i=1}^{m} d_i \leq s \leq -1). \tag{32}$$

The eveness of m gives one more zero of $W(s,\mathbf{d}^m)$ which arises from the parity properties of $V(s,\mathbf{d}^m)$, namely, $V(0,\mathbf{d}^{2k}) = 0$. The last equality immediately generates a zero $-\xi\{\mathbf{d}^{2k}\}$ of $W(s,\mathbf{d}^{2k})$ that together with (32) proves the first part (27) of Lemma 3.

The second part of Lemma 3 follows from (26) and from the first part of (27) because a set of integers $\{\mathbb{Z}/p\mathbb{Z}\}$ represents the zeroes of the periodic function $\Psi_p(s)$. ■

The complexity of the exponents sequence $\{\mathbf{d}^m\}$ and its large length make the calculative procedure of restoration of $Q_j^m(s)$ very cumbersome. Therefore it is important to find the inner properties of $\{\mathbf{d}^m\}$ when this procedure could be essentially reduced.

Lemma 4.4. *Let m-tuple $\{\mathbf{d}^m\} = \{d_1, d_2, ..., d_r, d_r, ..., d_m\}$ contains an exponent d_r twice. Then the Sylvester wave $V(s,\mathbf{d}^m)$ is related to the Sylvester wave $V(s,\mathbf{d}^{m_1})$ produced by the the non-degenerated tuple $\{\mathbf{d}^{m_1}\} = \{d_1, d_2, ..., d_r, .., .d_m, 2d_r\}$ as follows*

$$V(s,\mathbf{d}^m) = V(s - \frac{d_r}{2},\mathbf{d}^{m_1}) + V(s + \frac{d_r}{2},\mathbf{d}^{m_1})\;. \tag{33}$$

Proof. According to the definition (3)

$$(1+t^{d_r})\cdot\sum_s W(s,\mathbf{d}^{m_1})\cdot t^s=\sum_s W(s,\mathbf{d}^m)\cdot t^s\ .$$

Taking into account that $\xi\{\mathbf{d}^{m_1}\}-\xi\{\mathbf{d}^m\}=d_r/2$ and equating powers of t in the latter equation we obtain the stated relation (33) according to the definition (22). ■

We will make worth of relation (33) during the evaluation of the expression $V(s,\mathbf{d}^m)$ for the Coxeter group D_m.

5 Recursion formulas for $V(s,\mathbf{d}^m)$.

The shift (22) transforms the relation (8) into

$$V(s+\tau\{\mathbf{d}^m\},\mathbf{d}^m)=V(s,\mathbf{d}^m)+\sum_{p=0}^{\delta_m-1}V(s+\tau\{\mathbf{d}^m\}-\lambda_p\cdot d_m,\mathbf{d}^{m-1})\ ,\quad \lambda_p=p+\frac{1}{2} \tag{34}$$

and the relation (17) into

$$V(s,\mathbf{d}^m)=R_m^m(s)+\sum_{j=1}^{m-1}R_j^m(s)\cdot s^{m-j}\ , \tag{35}$$

where

$$R_j^m(s)=\sum_{i=1}^{j}C_{m-i}^{j-i}\cdot(-\xi\{\mathbf{d}^m\})^{j-i}\cdot Q_i^m(s-\xi\{\mathbf{d}^m\})\ ,$$

i.e., $R_1^m(s)=Q_1^m(s-\xi\{\mathbf{d}^m\})$; $R_2^m(s)=Q_2^m(s-\xi\{\mathbf{d}^m\})-(m-1)\cdot\xi\{\mathbf{d}^m\}\cdot Q_1^m(s-\xi\{\mathbf{d}^m\})$ etc. This means that the functions $R_j^m(s)$ and $Q_j^m(s)$ have the same period $\tau\{\mathbf{d}^j\}$.

Inserting the expansion (35) into the relation (34) and equating powers of s we can obtain for $k=1,2,\ldots,m-1$

$$\sum_{j=1}^{k}C_{m-j}^{m-1-k}\cdot R_j^m(s)\cdot\tau\{\mathbf{d}^m\}^{k+1-j}=\sum_{p=0}^{\delta_m-1}\sum_{j=1}^{k}R_j^{m-1}(s-\lambda_p\cdot d_m)\cdot C_{m-1-j}^{m-1-k}\cdot(\tau\{\mathbf{d}^m\}-\lambda_p\cdot d_m)^{k-j}. \tag{36}$$

For the first successive values of k the latter equation (36) gives

$$\begin{aligned}
R_1^m(s)&=\frac{1}{(m-1)\cdot\tau\{\mathbf{d}^m\}}\sum_{p=0}^{\delta_m-1}R_1^{m-1}(s-\lambda_p\cdot d_m)\ ,\\
R_2^m(s)&=\frac{1}{(m-2)\cdot\tau\{\mathbf{d}^m\}}\sum_{p=0}^{\delta_m-1}R_2^{m-1}(s-\lambda_p\cdot d_m)+\sum_{p=0}^{\delta_m-1}(\frac{1}{2}-\frac{\lambda_p}{\delta_m})\cdot R_1^{m-1}(s-\lambda_p\cdot d_m)\ ,\\
R_3^m(s)&=\frac{1}{(m-3)\cdot\tau\{\mathbf{d}^m\}}\sum_{p=0}^{\delta_m-1}R_3^{m-1}(s-\lambda_p\cdot d_m)+\sum_{p=0}^{\delta_m-1}(\frac{1}{2}-\frac{\lambda_p}{\delta_m})\cdot R_2^{m-1}(s-\lambda_p\cdot d_m)+\\
&\quad\frac{m-2}{2}\cdot\tau\{\mathbf{d}^m\}\sum_{p=0}^{\delta_m-1}(\frac{1}{6}-\frac{\lambda_p}{\delta_m}+\frac{\lambda_p^2}{\delta_m^2})\cdot R_1^{m-1}(s-\lambda_p\cdot d_m)\ .
\end{aligned} \tag{37}$$

It is easy to see that in the summands of the latter formulas (37) there appear the Bernoulli polynomials $\mathcal{B}_i(1-\frac{\lambda_p}{\delta_m})$: $\mathcal{B}_0(x)=1$, $\mathcal{B}_1(x)=x-1/2$, $\mathcal{B}_2(x)=x^2-x+1/6$, $\mathcal{B}_3(x)=x^3-3/2\ x^2+1/2\ x$, etc [1]. Continuing the evaluation of the general expression for $R_j^m(s),\ 1<j<m$, we arrive at

Lemma 5.1. *$R_j^m(s)$ for $1\le j<m$ is given by the formula*

$$R_j^m(s)=\frac{1}{m-j}\cdot\sum_{l=0}^{j-1}(\tau\{\mathbf{d}^m\})^{l-1}\cdot C_{m-1-j+l}^{l}\sum_{p=0}^{\delta_m-1}\mathcal{B}_l(1-\frac{\lambda_p}{\delta_m})\cdot R_{j-l}^{m-1}(s-\lambda_p\cdot d_m)\ . \tag{38}$$

Proof. Before going to the proof we recall two identities for the Bernoulli polynomials [1],[9]

$$\mathcal{B}_l(x+y)-\mathcal{B}_l(x)=\sum_{j=1}^{l}C_l^j\cdot y^j\cdot\mathcal{B}_{l-j}(x)\ ,\quad \mathcal{B}_l(1+x)-\mathcal{B}_l(x)=lx^{l-1}\ . \tag{39}$$

Using the definition (35) we check that formula (38) satisfies (34).

$$\begin{aligned}
&V(s,\mathbf{d}^m)=R_m^m(s)+\sum_{j=1}^{m-1}s^j\sum_{l=j}^{m-1}C_l^j\frac{(\tau\{\mathbf{d}^m\})^{l-j-1}}{l}\sum_{p=0}^{\delta_m-1}\mathcal{B}_{l-j}(1-\frac{\lambda_p}{\delta_m})R_{m-l}^{m-1}(s-\lambda_p d_m)=\\
&R_m^m(s)+\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)\sum_{j=1}^{l}C_l^j\left(\frac{s}{\tau\{\mathbf{d}^m\}}\right)^j\mathcal{B}_{l-j}(1-\frac{\lambda_p}{\delta_m})=\\
&R_m^m(s)+\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)\left[\mathcal{B}_l(1+\frac{s-\lambda_p d_m}{\tau\{\mathbf{d}^m\}})-\mathcal{B}_l(1-\frac{\lambda_p}{\delta_m})\right],
\end{aligned}\tag{40}$$

where we use the first of the identities (39). Having in mind the $\tau\{\mathbf{d}^m\}$-periodicity of functions $R_j^m(s)$ and $R_j^{m-1}(s)$ and the second identity (39) we may rewrite the difference in the l.h.s of relation (34) in the following form:

$$\begin{aligned}
&V(s,\mathbf{d}^m)-V(s-\tau\{\mathbf{d}^m\},\mathbf{d}^m)= \\
&\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)\left[\mathcal{B}_l(1-\frac{\lambda_p}{\delta_m}+\frac{s}{\tau\{\mathbf{d}^m\}})-\mathcal{B}_l(-\frac{\lambda_p}{\delta_m}+\frac{s}{\tau\{\mathbf{d}^m\}})\right]\\
&\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)l\left(\frac{s-\lambda_p d_m}{\tau\{\mathbf{d}^m\}}\right)^{l-1}=\\
&\sum_{p=0}^{\delta_m-1}\sum_{l=0}^{m-2}(s-\lambda_p d_m)^l R_{m-1-l}^{m-1}(s-\lambda_p d_m)=\sum_{p=0}^{\delta_m-1}V(s-\lambda_p d_m,\mathbf{d}^{m-1}).\qquad\blacksquare
\end{aligned}\tag{41}$$

The formula (38) enables to restore all terms $R_k^m(s)$ except the last $R_m^m(s)$. Actually we can learn about it from the following consideration. Let us separate $R_{m-k}^m(s)$ in the following way

$$R_{m-k}^m(s)=\mathcal{R}_{m-k}^m(s)+r_{m-k}^m(s)\ ,\quad 0\le k\le m-1\ , \tag{42}$$

where

$$\mathcal{R}_{m-k}^{m}(s) \;=\; \sum_{l=1}^{m-k-1} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l+k} \cdot C_{l+k}^{k} \sum_{p=0}^{\delta_m-1} \mathcal{B}_l(1-\frac{\lambda_p}{\delta_m}) \cdot R_{m-k-l}^{m-1}(s-\lambda_p \cdot d_m) \tag{43}$$

$$r_{m-k}^{m}(s) \;=\; \frac{1}{k\cdot\tau\{\mathbf{d}^m\}} \sum_{p=0}^{\delta_m-1} R_{m-k}^{m-1}(s-\lambda_p d_m)\ ,\ r_{m-k}^{m}(s)=r_{m-k}^{m}(s-d_m)\ ,\ (k\neq 0) \tag{44}$$

The representation (42) and d_m-periodicity of the function $r_{m-k}^{m}(s)$ make possible to prove the following

Lemma 5.2. *$R_{m-k}^{m}(s)$ for $0 \le k \le m-1$ and $\mathcal{R}_{m-k}^{m}(s)$ for $0 < k \le m-1$ satisfy the recursion relation*

$$R_{m-k}^{m}(s) - R_{m-k}^{m}(s-d_m) = \mathcal{R}_{m-k}^{m}(s) - \mathcal{R}_{m-k}^{m}(s-d_m) =$$
$$\sum_{j=k+1}^{m-1} \left\{ (-d_m)^{j-k} \cdot C_j^k \cdot R_{m-j}^{m}(s-d_m) + (-\frac{d_m}{2})^{j-1-k} \cdot C_{j-1}^{k} \cdot R_{m-j}^{m-1}(s-\frac{d_m}{2}) \right\}. \tag{45}$$

Proof. Inserting (35) into (24), expanding the powers of binomials into sums and equating the powers of s in the latter equation we obtain the relation (45) for the function $R_{m-k}^{m}(s)$, $0 \le k \le m-1$. Using the definition (42) we immediately arrive at the relation for the function $\mathcal{R}_{m-k}^{m}(s)$, $0 < k \le m-1$. ■

In the special case $k=0$ the general relation (45) produces the recursion for $R_m^m(s)$

$$R_m^m(s) - R_m^m(s-d_m) = \sum_{j=1}^{m-1} \left\{ (-d_m)^j \cdot R_{m-j}^{m}(s-d_m) + (-\frac{d_m}{2})^{j-1} \cdot R_{m-j}^{m-1}(s-\frac{d_m}{2}) \right\}. \tag{46}$$

We can not use directly (43) for $k=0$ since $r_m^m(s)$ can not be derived from (44). But it is a good mathematical intuition to exploit the formula (43) for $k=0$ in order to prove

Lemma 5.3. *$\mathcal{R}_m^m(s)$ is given by the formula*

$$\mathcal{R}_m^m(s) = \sum_{l=1}^{m-1} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l} \sum_{p=0}^{\delta_m-1} \mathcal{B}_l(1-\frac{\lambda_p}{\delta_m}) \cdot R_{m-l}^{m-1}(s-\lambda_p \cdot d_m)\ . \tag{47}$$

Proof. In order to prove that $\mathcal{R}_m^m(s)$ given by (47) satisfies the difference equation (46) we consider a difference $\mathcal{R}_m^m(s) - \mathcal{R}_m^m(s-d_m) = \Delta_m(s) = \Delta_m^1(s) + \Delta_m^2(s)$:

$$\Delta_m(s) = \sum_{l=1}^{m-1} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l} \sum_{p=0}^{\delta_m-1} \mathcal{B}_l(1-\frac{\lambda_p}{\delta_m}) \cdot \left[R_{m-l}^{m-1}(s-\lambda_p d_m) - R_{m-l}^{m-1}(s-\lambda_{p+1} d_m) \right]$$

with

$$\Delta_m^1(s) = \sum_{l=1}^{m-1} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l} \left\{ \mathcal{B}_l(1-\frac{1}{2\delta_m}) - \mathcal{B}_l(-\frac{1}{2\delta_m}) \right\} \cdot R_{m-l}^{m-1}(s-\frac{d_m}{2})\ ,$$

$$\Delta_m^2(s) = \sum_{l=1}^{m-1} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l} \sum_{p=1}^{\delta_m} \left\{ \mathcal{B}_l(1-\frac{\lambda_p}{\delta_m}) - \mathcal{B}_l(1-\frac{\lambda_p}{\delta_m}+\frac{1}{\delta_m}) \right\} \cdot R_{m-l}^{m-1}(s-\lambda_p d_m)\ .$$

The first term $\Delta_m^1(s)$ is calculated with the help of one of the identities (39):

$$\Delta_m^1(s) = \sum_{l=1}^{m-1}(-\frac{d_m}{2})^{l-1} \cdot R_{m-l}^{m-1}(s-\frac{d_m}{2}) \ . \tag{48}$$

Using another identity from (39) we may write for $\Delta_m^2(s)$:

$$\Delta_m^2(s) = \sum_{l=1}^{m-1}\sum_{j=1}^{l}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\cdot C_l^j\cdot(-\frac{1}{\delta_m})^j\sum_{p=1}^{\delta_m}\mathcal{B}_{l-j}(1-\frac{\lambda_{p-1}}{\delta_m})\cdot R_{m-l}^{m-1}(s-\lambda_p d_m) \ .$$

Changing here summation order $\sum_{k=l+1}^{m-1}\sum_{j=l+1}^{k} = \sum_{j=l+1}^{m-1}\sum_{k=j}^{m-1}$ and comparing the inner sum with (38) we arrive at

$$\Delta_m^2(s) = \sum_{j=1}^{m-1}(-d_m)^j \cdot R_{m-j}^m(s-d_m) \tag{49}$$

Then (48) and (49) prove the Lemma. ■

From this Lemma follows an existence of d_m-periodic function $r_m^m(s) = r_m^m(s-d_m)$ which could not be derived from (44). Unknown function $r_m^m(s)$ corresponds to vanishing harmonics in the r.h.s. of equation (45). We are free to choose any basic system of continuous $\tau\{\mathbf{d}^m\}$-periodic functions. This arbitrariness can affect behaviour of $W(s,\mathbf{d}^m)$ only for non-integer s that does not violate the recursion relation (5). In the rest of the paper we will choose a basic system of the simplest periodic functions sin and cos.

The function $r_m^m(s)$ corresponds to the harmonics of the type

$$\left\{\begin{array}{c}\sin\\ \cos\end{array}\right\}\frac{2\pi n}{d_m}s$$

Because the parity properties of $R_m^m(s)$ coincide with that of $V(s,\mathbf{d}^m)$ itself we can rewrite (35) in the following form

$$V(s,\mathbf{d}^{2m}) = \sum_{j=1}^{2m-1} R_j^{2m}(s)\cdot s^{2m-j} + \mathcal{R}_{2m}^{2m}(s) + \sum_n \rho_n^{2m}\cdot\sin\frac{2\pi n}{d_{2m}}s \ , \tag{50}$$

$$V(s,\mathbf{d}^{2m+1}) = \sum_{j=1}^{2m} R_j^{2m+1}(s)\cdot s^{2m+1-j} + \mathcal{R}_{2m+1}^{2m+1}(s) + \sum_n \rho_n^{2m+1}\cdot\cos\frac{2\pi n}{d_{2m+1}}s \ . \tag{51}$$

In order to produce $r_m^m(s)$ we use some of zeroes $\mathfrak{s}$, described in the preceding Section, constructing a system of linear equations for $[(m+1)/2]$ coefficients ρ_n; n runs from 1 to $m/2$ in (50 and from 0 to $(m-1)/2$ in (51). We use a trivial identity $V(\xi(\mathbf{d}^m),\mathbf{d}^m)=1$, and choose the values of s out of the set $\mathfrak{s}$, adding homogeneous equations to arrive at a non-degenerate inhomogeneous system of linear equations. This system is solved further to produce the final expression for corresponding Sylvester wave. These explicit expressions are given in the next Section. Appendix B presents two instructive examples of the above procedure.

6 Sylvester waves $V(s,G)$.

We start with the symmetric group $\mathcal{S}_m$ because of two reasons: first, of their relation with restricted partition numbers and , second, they arranged natural basis to utilize the Sylvester waves $V(s,G)$ in all Coxeter groups.

6.1 Symmetric groups $\mathcal{S}_m$.

Making use of the procedure developed in the previous section we present here first ten Sylvester waves $V(s,\mathcal{S}_m)$, $m=1,...,10.$ [2]

$\underline{G=\mathcal{S}_m}$, $d_r=1,2,3,...,m$, $\xi(\mathcal{S}_m)=\frac{m(m+1)}{4}$,

$$
\begin{aligned}
V(s,\mathcal{S}_1) &= 1,\\
V(s,\mathcal{S}_2) &= \frac{s}{2}-\frac{1}{4}\sin\pi s,\\
V(s,\mathcal{S}_3) &= \frac{s^2}{12}-\frac{7}{72}-\frac{1}{8}\cos\pi s+\frac{2}{9}\cos\frac{2\pi s}{3},\\
V(s,\mathcal{S}_4) &= \frac{s^3}{144}-\frac{s}{96}\cdot(5+3\cos\pi s)+\frac{1}{8}\sin\frac{\pi s}{2}-\frac{2}{9\sqrt{3}}\sin\frac{2\pi s}{3},\\
V(s,\mathcal{S}_5) &= \frac{s^4}{2880}-\frac{11\cdot s^2}{1152}-\frac{s}{64}\cdot\sin\pi s+\frac{17083}{691200}-\frac{2}{27}\cos\frac{2\pi s}{3}+\\
&\quad \frac{1}{8\sqrt{2}}\cos\frac{\pi s}{2}+\frac{2}{25}(-\cos\frac{2\pi s}{5}+\cos\frac{4\pi s}{5}),\\
V(s,\mathcal{S}_6) &= \frac{s^5}{86400}-\frac{91\cdot s^3}{103680}+\frac{s^2}{768}\cdot\sin\pi s+\frac{s}{829440}\cdot(9191-10240\cos\frac{2\pi s}{3})-\\
&\quad \frac{161}{9216}\sin\pi s-\frac{1}{16\sqrt{2}}\sin\frac{\pi s}{2}-\frac{1}{81\sqrt{3}}\sin\frac{2\pi s}{3}-\frac{1}{18}\sin\frac{\pi s}{3}-\\
&\quad \frac{2}{25\sqrt{5}}(\sin\frac{\pi}{5}\sin\frac{4\pi s}{5}+\sin\frac{2\pi}{5}\sin\frac{2\pi s}{5}),\\
V(s,\mathcal{S}_7) &= \frac{s^6}{3628800}-\frac{s^4}{20736}+\frac{s^2}{38400}\cdot(71+25\cos\pi s)-\frac{s}{81\sqrt{3}}\cdot\sin\frac{2\pi s}{3}-\\
&\quad \frac{52705}{6096384}-\frac{77}{4608}\cos\pi s-\frac{1}{32}\cos\frac{\pi s}{2}-\frac{5}{486}\cos\frac{2\pi s}{3}-\frac{1}{18}\cos\frac{\pi s}{3}+\\
&\quad \frac{2}{25\sqrt{5}}(\cos\frac{2\pi s}{5}-\cos\frac{4\pi s}{5})+\frac{2}{49}(\cos\frac{2\pi s}{7}+\cos\frac{4\pi s}{7}+\cos\frac{6\pi s}{7}),\\
V(s,\mathcal{S}_8) &= \frac{s^7}{203212800}-\frac{17\cdot s^5}{9676800}+\frac{s^3}{8294400}\cdot(1343+225\cos\pi s)+\\
&\quad s\cdot(-\frac{16133}{4976640}-\frac{1}{256}\cos\frac{\pi s}{2}+\frac{1}{243}\cos\frac{2\pi s}{3}-\frac{31}{12288}\cos\pi s)+\\
&\quad \frac{1}{32}(\sin\frac{\pi s}{4}-\sin\frac{3\pi s}{4})-\frac{1}{128}\sin\frac{\pi s}{2}+\frac{1}{162\sqrt{3}}\sin\frac{2\pi s}{3}+\frac{1}{18\sqrt{3}}\sin\frac{\pi s}{3}+\\
&\quad \frac{4}{125}(\sin\frac{2\pi}{5}\sin\frac{4\pi s}{5}-\sin\frac{\pi}{5}\sin\frac{2\pi s}{5})-
\end{aligned}
\tag{52}
$$

[2]Having in mind the results of Sylvester [10],[11] and Glaisher [7] for restricted partition numbers for $m\le 9$ we repeat them adding a formula for $m=10$. The list of $V(s,\mathcal{S}_m)$ can be simply continued up to any finite m with the help of the symbolic code written in *Mathematica* language [14].

$$
\begin{aligned}
&\frac{1}{49}(\sin\frac{2\pi s}{7}\csc\frac{\pi}{7}-\sin\frac{4\pi s}{7}\csc\frac{2\pi}{7}+\sin\frac{6\pi s}{7}\csc\frac{3\pi}{7}),\\
V(s,\mathcal{S}_9) \;=\; &\frac{s^8}{14631321600}-\frac{19\cdot s^6}{418037760}+\frac{145597\cdot s^4}{16721510400}+\frac{s^3}{73728}\cdot\sin\pi s-\\
&s^2\cdot(\frac{67293991}{140460687360}+\frac{1}{4374}\cos\frac{2\pi s}{3})-\\
&s\cdot(\frac{1}{256\sqrt{2}}\sin\frac{\pi s}{2}+\frac{1}{1458\sqrt{3}}\sin\frac{2\pi s}{3}+\frac{205}{98304}\sin\pi s)+\frac{199596951167}{56184274944000}+\\
&\frac{1}{64}(\cos\frac{\pi s}{4}\csc\frac{\pi}{8}-\cos\frac{3\pi s}{4}\csc\frac{3\pi}{8})+\frac{2}{125}(\cos\frac{4\pi s}{5}-\cos\frac{2\pi s}{5})-\\
&\frac{5}{512\sqrt{2}}\cos\frac{\pi s}{2}+\frac{257}{17496}\cos\frac{2\pi s}{3}+\frac{1}{36\sqrt{3}}\cos\frac{\pi s}{3}+\\
&\frac{2}{81}(-\cos\frac{2\pi s}{9}+\cos\frac{4\pi s}{9}+\cos\frac{8\pi s}{9})-\\
&\frac{1}{98}(\cos\frac{2\pi s}{7}\csc\frac{\pi}{7}\csc\frac{2\pi}{7}+\cos\frac{4\pi s}{7}\csc\frac{2\pi}{7}\csc\frac{3\pi}{7}+\cos\frac{6\pi s}{7}\csc\frac{3\pi}{7}\csc\frac{\pi}{7}),\\
V(s,\mathcal{S}_{10}) \;=\; &\frac{s^9}{1316818944000}-\frac{11\cdot s^7}{12541132800}+\frac{113113\cdot s^5}{358318080000}-\frac{\sin\pi s}{2949120}\cdot s^4-\\
&\frac{18063859\cdot s^3}{468202291200}+s^2\cdot(\frac{1}{4374\sqrt{3}}\sin\frac{2\pi s}{3}+\frac{143}{1179648}\sin\pi s)+\\
&s\cdot[\frac{273512277643}{240789749760000}+\frac{1}{512\sqrt{2}}\cos\frac{\pi s}{2}+\frac{7}{13122}\cos\frac{2\pi s}{3}+\\
&\frac{1}{625}(\cos\frac{4\pi s}{5}-\cos\frac{2\pi s}{5})]-\frac{2877523}{707788800}\sin\pi s-\frac{1211}{52488\sqrt{3}}\sin\frac{2\pi s}{3}-\\
&\frac{5}{1024\sqrt{2}}\sin\frac{\pi s}{2}-\frac{1}{108}\sin\frac{\pi s}{3}+\frac{1}{64\sqrt{2}}(\csc\frac{3\pi}{8}\sin\frac{3\pi s}{4}-\csc\frac{\pi}{8}\sin\frac{\pi s}{4})+\\
&\frac{1}{50}(\sin\frac{3\pi s}{5}-\sin\frac{\pi s}{5})-\frac{2\sqrt{2}}{625}(\frac{\sqrt{5}+2}{\sqrt{5+\sqrt{5}}}\sin\frac{2\pi s}{5}+\frac{\sqrt{5}-2}{\sqrt{5-\sqrt{5}}}\sin\frac{4\pi s}{5})-\\
&\frac{1}{196}\csc\frac{\pi}{7}\csc\frac{2\pi}{7}\csc\frac{3\pi}{7}(\sin\frac{6\pi s}{7}+\sin\frac{4\pi s}{7}-\sin\frac{2\pi s}{7})+\\
&\frac{1}{81}(\csc\frac{4\pi}{9}\sin\frac{8\pi s}{9}+\csc\frac{2\pi}{9}\sin\frac{4\pi s}{9}+\csc\frac{\pi}{9}\sin\frac{2\pi s}{9}).
\end{aligned}
$$

6.2 Coxeter groups.

Let us define two auxiliary functions

$$
\begin{aligned}
U_+(s,p,G) &= V(s+p,G)+V(s-p,G)\;,\\
U_-(s,p,G) &= V(s+p,G)-V(s-p,G)
\end{aligned}
\qquad (53)
$$

with obvious properties

$$
\begin{aligned}
U_+(s,p,\mathbf{d}^m/d_r) &= U_-(s,p+\frac{d_r}{2},\mathbf{d}^m)-U_-(s,p-\frac{d_r}{2},\mathbf{d}^m)\;,\;\; U_+(s,0,G)=2V(s,G),\\
U_-(s,p,\mathbf{d}^m/d_r) &= U_+(s,p+\frac{d_r}{2},\mathbf{d}^m)-U_+(s,p-\frac{d_r}{2},\mathbf{d}^m)\;,\;\; U_-(s,\frac{d_r}{2},\mathbf{d}^m)=V(s,\mathbf{d}^m/d_r),
\end{aligned}
$$

where $(m-1)$-*tuple* $\{\mathbf{d}^m/d_r\}=\{d_1,d_2,...,d_{r-1},d_{r+1},...,d_m\}$ doesn't contain d_r-exponent.

Sylvester waves for the Coxeter groups are given below expressed through the relations elaborated in the previous Sections.

$\underline{G = A_m}$, $d_r = 2, 3, ..., m+1$; $\xi(A_m) = \frac{1}{4}m(m+3)$

$$V(s, A_m) = U_-(s, \frac{1}{2}, \mathcal{S}_m). \tag{54}$$

$\underline{G = B_m}$, $d_r = 2, 4, 6, ..., 2m$; $\xi(B_m) = \frac{1}{2}m(m+1)$

$$V(s, B_m) = \frac{1}{2}\Psi_2(s - \xi(B_m)) \cdot U_+(\frac{s}{2}, 0, \mathcal{S}_m) \ . \tag{55}$$

In the list for D_m groups the degree m occurs twice when m is even. This is the only case involving such a repetition.

$\underline{G = D_m}$, $d_r = 2, 4, 6, ..., 2(m-1), m$, $m \geq 3$; $\xi(D_m) = \frac{1}{2}m^2$,

$$\begin{aligned} V(s, D_{2m}) &= \Psi_2(s) \cdot U_+(\frac{s}{2}, \frac{m}{2}, \mathcal{S}_{2m}), \qquad (56) \\ V(s, D_{2m+1}) &= \sum_{s_1=0}^{s-\xi(D_{2m+1})} V(s + \frac{2m+1}{2} - s_1, B_{2m}) \cdot \Psi_{2m+1}(s_1), \\ V(s, D_3) &= V(s, A_3), \\ V(s, D_5) &= U_-(s, \frac{11}{2}, \mathcal{S}_8) - U_-(s, \frac{9}{2}, \mathcal{S}_8) - U_-(s, \frac{5}{2}, \mathcal{S}_8) + U_-(s, \frac{3}{2}, \mathcal{S}_8). \end{aligned}$$

$\underline{G = G_2}$, $d_r = 2, 6$; $\xi(G_2) = 4$,

$$V(s, G_2) = \Psi_2(s) \cdot U_-(\frac{s}{2}, 1, \mathcal{S}_3). \tag{57}$$

$\underline{G = F_4}$, $d_r = 2, 6, 8, 12$; $\xi(F_4) = 14$,

$$V(s, F_4) = \Psi_2(s) \cdot [\ U_+(\frac{s}{2}, \frac{7}{2}, \mathcal{S}_6) - U_+(\frac{s}{2}, \frac{3}{2}, \mathcal{S}_6)\ ]. \tag{58}$$

$\underline{G = E_6}$, $d_r = 2, 5, 6, 8, 9, 12$; $\xi(E_6) = 21$,

$$\begin{aligned} V(s, E_6) &= U_+(s, 18, \mathcal{S}_{12}) - U_+(s, 17, \mathcal{S}_{12}) - U_+(s, 15, \mathcal{S}_{12}) + \\ &\quad U_+(s, 13, \mathcal{S}_{12}) + U_+(s, 5, \mathcal{S}_{12}) - U_+(s, 2, \mathcal{S}_{12}) \ . \qquad (59) \end{aligned}$$

$\underline{G = E_7}$, $d_r = 2, 6, 8, 10, 12, 14, 18$; $\xi(E_7) = 35$,

$$V(s, E_7) = \Psi_2(s-1) \cdot [\ U_+(\frac{s}{2}, 5, \mathcal{S}_9) - U_+(\frac{s}{2}, 3, \mathcal{S}_9)\ ] \ . \tag{60}$$

$\underline{G = E_8}$, $d_r = 2, 8, 12, 14, 18, 20, 24, 30$; $\xi(E_8) = 64$,

$$\begin{aligned} V(s, E_8) &= \Psi_2(s) \cdot [\ U_-(\frac{s}{2}, 28, \mathcal{S}_{15}) + U_-(\frac{s}{2}, 21, \mathcal{S}_{15}) + U_-(\frac{s}{2}, 12, \mathcal{S}_{15}) + \\ &\quad U_-(\frac{s}{2}, 11, \mathcal{S}_{15}) - U_-(\frac{s}{2}, 8, \mathcal{S}_{15}) - U_-(\frac{s}{2}, 7, \mathcal{S}_{15}) - \\ &\quad U_-(\frac{s}{2}, 6, \mathcal{S}_{15}) - U_-(\frac{s}{2}, 26, \mathcal{S}_{15}) - U_-(\frac{s}{2}, 25, \mathcal{S}_{15})\ ]. \qquad (61) \end{aligned}$$

$\underline{G = H_3}$, $d_r = 2, 6, 10$; $\xi(H_3) = 9$,

$$V(s, H_3) = \Psi_2(s-1) \cdot [\, U_+(\frac{s}{2}, 3, \mathcal{S}_5) - U_+(\frac{s}{2}, 1, \mathcal{S}_5)\,]. \tag{62}$$

$\underline{G = H_4}$, $d_r = 2, 12, 20, 30$; $\xi(H_3) = 32$,

$$\begin{aligned} V(s, H_4) &= U_+(s, 32, E_8) - U_+(s, 24, E_8) - U_+(s, 18, E_8) - U_+(s, 14, E_8) + \\ &\quad U_+(s, 10, E_8) - U_+(s, 8, E_8) + U_+(s, 6, E_8) + U_+(s, 0, E_8). \end{aligned} \tag{63}$$

$\underline{G = I_m}$, $d_r = 2, m$; $\xi(I_m) = 1 + \frac{1}{2}m$

$$V(s, I_m) = \sum_{s_1=0}^{s-\xi(I_m)} \Psi_2(s - \xi(I_m) - s_1) \cdot \Psi_m(s_1) \ , \tag{64}$$

$$\begin{aligned} V(s, I_2) &= V(s, B_1) \ , \ \ V(s, I_3) = V(s, A_2) \ , \ \ V(s, I_4) = V(s, B_2) \ , \\ V(s, I_5) &= U_+(s, \frac{7}{2}, A_4) - U_+(s, \frac{1}{2}, A_4) \ , \\ V(s, I_6) &= V(s, G_2) \ , \ \ V(s, I_8) = U_+(s, 5, B_4) - U_+(s, 1, B_4) \\ V(s, I_{10}) &= U_-(s, 3, H_3) \ , \ \ V(s, I_{12}) = U_+(s, 7, F_4) - U_+(s, 1, F_4) \ . \end{aligned}$$

7 Acknowledgement

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A Asymptotic behaviour of lcm(1,2,...,N).

The arithmetical function *least common multiple* $\mathsf{lcm}(1,2,...,N) = \mathcal{L}(N)$ of the series of the natural numbers takes a specific place among the other arithmetical functions. It can neither be represented as the Cauchy integral of the generating function with subsequent evaluation with Hardy-Ramanujan circle method like different partition functions $p(N)$, $q(N)$, nor has it its genesis in Riemann's Zeta-function like many arithmetical functions $\mu(N)$, $\nu(N)$, $\phi(N)$, $d(N)$. $\mathcal{L}(N)$ appears naturally in the theory of restricted partition numbers as periods of Sylvester waves in symmetric groups $\mathcal{S}_N$.

Numerical calculations of $\frac{1}{N}\ln[\mathcal{L}(N)]$ in the range $0 < N < 550 \times 10^3$ give an oscillating behaviour around 1 with asymptotic approach to this value (Fig. 1). This enabled us to

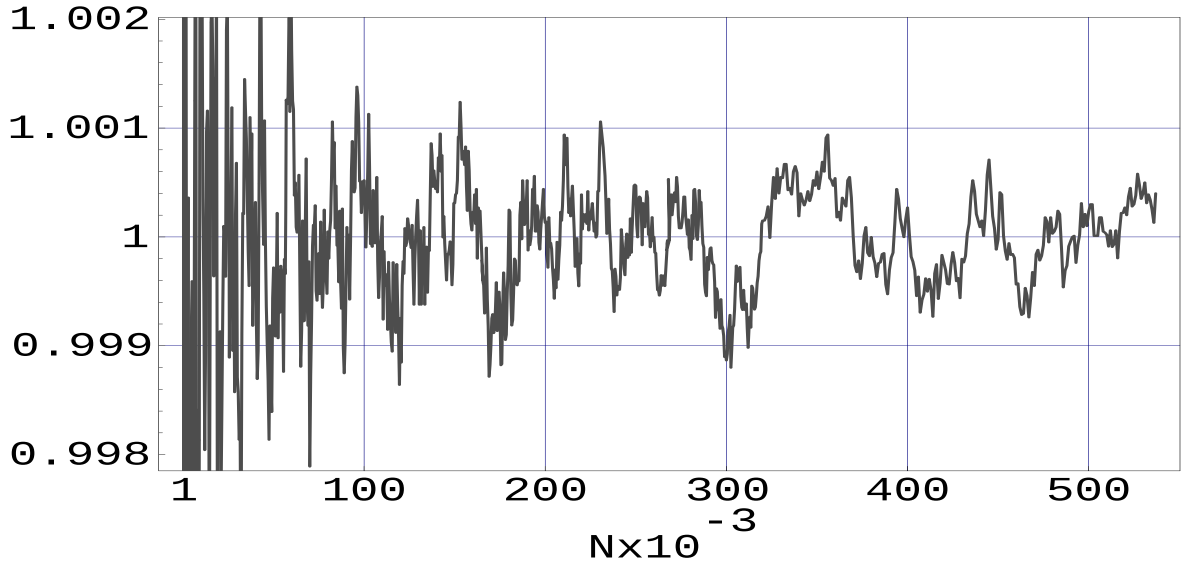


Figure 1: Asymptotic behaviour of $\frac{1}{N}\ln[\mathcal{L}(N)]$.

conjecture an asymptotic law

$$\lim_{N\to\infty} \frac{\ln \mathcal{L}(N)}{N} = 1. \tag{A1}$$

In the rest of this Appendix we give a proof of this statement. Before going to the proof we recall some facts of the prime number theory:

F1. The Prime Number Theorem (PNT)

$$\text{if} \quad \pi(N) = \sum_{p_i \le N} 1 \quad , \qquad \text{then} \quad \pi(N) \overset{N\to\infty}{\simeq} \frac{N}{\ln N} \; . \tag{A2}$$

where a sum is running over all primes p_i up to N.

F2. Let us set after Chebyshev

$$\theta(N) = \sum_{p_i \le N} \ln p_i \; , \tag{A3}$$

then PNT is equivalent to $\theta(N) \overset{N\to\infty}{\simeq} N$.

F3. The Rieman hypothesis is equivalent to

$$\theta(N) = N + O(\sqrt{N}\ln N) \tag{A4}$$

Now it follows

Lemma A.

$$\lim_{N\to\infty} \frac{\ln \mathcal{L}(N)}{N} = 1.$$

and assuming the Rieman hypothesis

$$\ln \mathcal{L}(N) = N + O(\sqrt{N}\ln N).$$

Proof of Lemma A.

We write the prime decomposition of $\mathcal{L}(N)$ as $\mathcal{L}(N) = \prod p^{k_p}$. Clearly , for a prime to divide $\mathcal{L}(N)$ it has to be at most less than N. Moreover the highest k power of p dividing one of the integers 1, 2, ..., N is

$$k_p = [\frac{\ln N}{\ln p}]$$

Thus we find

$$\ln \mathcal{L}(N) = \sum_{p_i \le N} [\frac{\ln N}{\ln p}] \cdot \ln p \tag{A5}$$

To estimate $\ln \mathcal{L}(N)$, break the previous sum into two parts, one Q_1 coming from primes $p \le \sqrt{N}$ and the second Q_2 from primes $\sqrt{N} \le p \le N$:

$$Q_1 = \sum_{p_i \le \sqrt{N}} [\frac{\ln N}{\ln p}] \cdot \ln p\ , \quad Q_2 = \sum_{\sqrt{N} \le p \le N} [\frac{\ln N}{\ln p}] \cdot \ln p$$

For estimating Q_1, use $[x] \le x$ and so we find

$$Q_1 \le \sum_{p_i \le \sqrt{N}} \frac{\ln N}{\ln p} \cdot \ln p = \ln N \cdot \pi(\sqrt{N}) \simeq 2\sqrt{N}$$

by the PNT. For the second sum Q_2 note that if $\sqrt{N} \le p \le N$ then

$$1 \le \frac{\ln N}{\ln p} < 2$$

and hence its integer part is identically 1. Thus

$$Q_2 = \sum_{\sqrt{N} \le p \le N} 1 \cdot \ln p = \theta(N) - \theta(\sqrt{N})$$

Since $\theta(\sqrt{N}) \simeq \sqrt{N}$, we obtain finally

$$\ln \mathcal{L}(N) = \theta(N) + \theta(\sqrt{N})$$

Our Lemma A follows immediately from F2, F3. ■

B Derivation of Sylvester waves $V(s,\mathcal{S}_4)$ and $V(s,\mathcal{S}_5)$.

We will illustrate how do the formulas (38-51) work in the case of the symmetric groups $\mathcal{S}_4$ and $\mathcal{S}_5$.

We start with Sylvester wave $V(s,\mathcal{S}_3)$ taken from (52)

$$V(s,\mathcal{S}_3)=\frac{s^2}{12}-\frac{7}{72}-\frac{1}{8}\cos\pi s+\frac{2}{9}\cos\frac{2\pi s}{3} \tag{B1}$$

and with successive usage of the formulas (38) and (47) one can obtain

$$R_1^4(s)=\frac{1}{144}\;,\quad R_2^4(s)=0\;,\quad R_3^4(s)=-\frac{1}{96}\cdot(5+3\cos\pi s)\;,\quad \mathcal{R}_4^4(s)=-\frac{2}{9\sqrt{3}}\sin\frac{2\pi s}{3}\;. \tag{B2}$$

Now we will use the representation (50)

$$V(s,\mathcal{S}_4)=\sum_{j=1}^{3}R_j^4(s)\cdot s^{4-j}+\mathcal{R}_4^4(s)+\rho_1^4\cdot\sin\frac{\pi}{2}s+\rho_2^4\cdot\sin\pi s\;. \tag{B3}$$

Since $V(s,\mathcal{S}_4)=W(s-5,\mathcal{S}_4)$ the variable s takes only integer values what makes the last contribution in (B3) into the $V(s,\mathcal{S}_4)$ irrelevant. The unknown coefficient ρ_1^4 is determined with help of *zeroes* (27) of $W(s,\mathcal{S}_4)$

$$0=V(1,\mathcal{S}_4)=\sum_{j=1}^{3}R_j^4(1)+\mathcal{R}_4^4(1)+\rho_1^4\;,\quad\text{or}\quad \rho_1^4=\frac{1}{8} \tag{B4}$$

Thus we arrive at the Sylvester wave $V(s,\mathcal{S}_4)$ presented in (52).

Repeating the same procedure with symmetric group $\mathcal{S}_5$ we find

$$\begin{aligned} R_1^5(s)&=\frac{1}{2880}\;,\quad R_2^5(s)=0\;,\quad R_3^5(s)=-\frac{11}{1152}\;,\quad R_4^5(s)=-\frac{1}{64}\sin\pi s\;,\\ \mathcal{R}_5^5(s)&=\frac{475}{27648}-\frac{2}{27}\cos\frac{2\pi s}{3}+\frac{1}{8\sqrt{2}}\cos\frac{\pi s}{2}\;. \end{aligned} \tag{B5}$$

The representation (51) produces

$$V(s,\mathcal{S}_5)=\sum_{j=1}^{4}R_j^5(s)\cdot s^{5-j}+\mathcal{R}_5^5(s)+\rho_0^5+\rho_1^5\cdot\cos\frac{2\pi s}{5}+\rho_2^5\cdot\cos\frac{4\pi s}{5}\;. \tag{B6}$$

Since $V(s,\mathcal{S}_5)=W(s-\frac{15}{2},\mathcal{S}_5)$ the variable s has only half-integer values. By solving three linear equations $V(\frac{1}{2},\mathcal{S}_5)=V(\frac{3}{2},\mathcal{S}_5)=V(\frac{5}{2},\mathcal{S}_5)=0$ we find

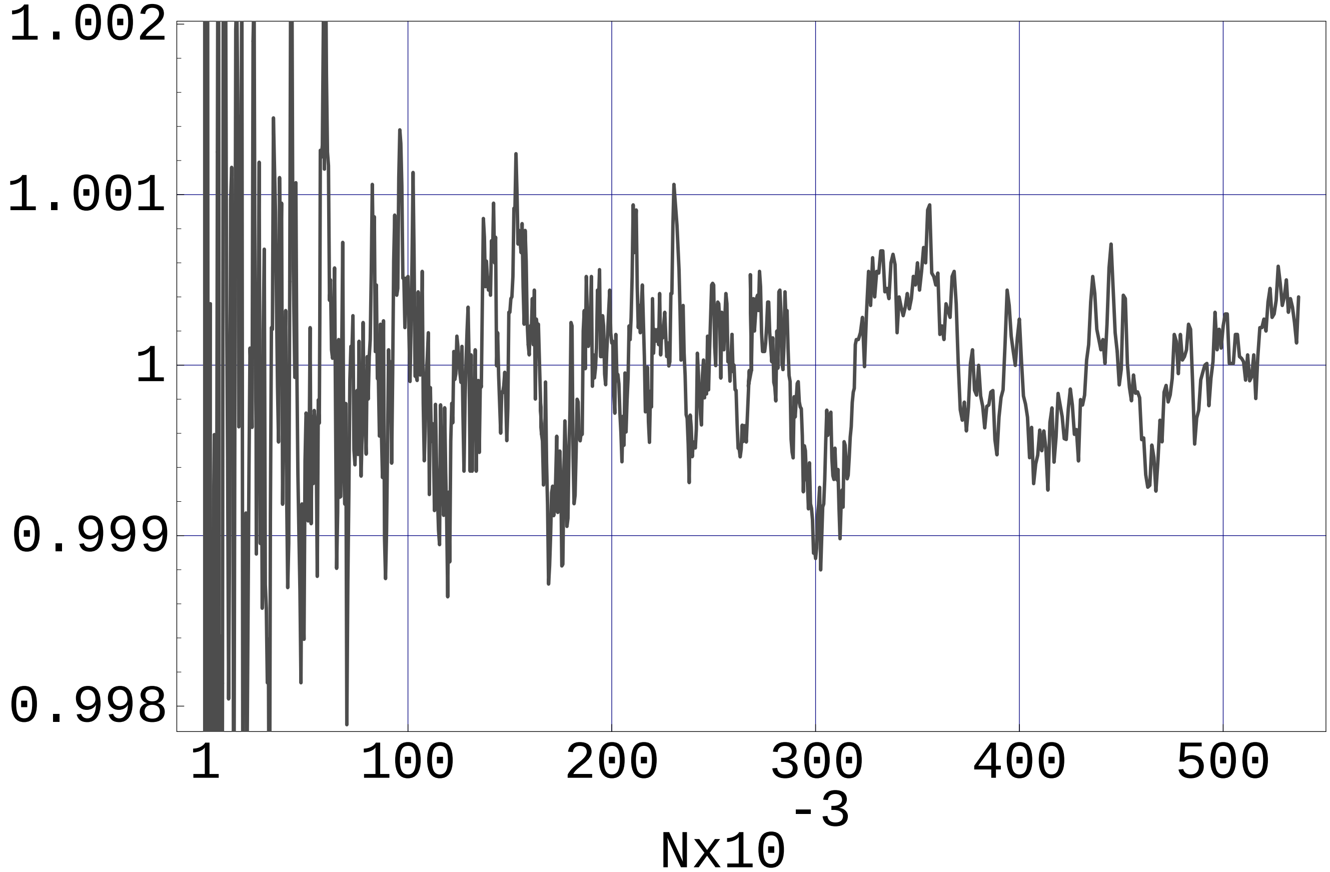
$$\rho_0^5=\frac{217}{28800}\;,\quad \rho_1^5=-\frac{2}{25}\;,\quad \rho_2^5=\frac{2}{25}\;, \tag{B7}$$

which together with (B6) produces the Sylvester wave $V(s,\mathcal{S}_5)$ from (52).

C Table of restricted partition numbers $W(s, \mathcal{S}_m)$.

In this Appendix we give the Table of the restricted partition numbers $\mathcal{P}_m(s) = W(s, \mathcal{S}_m)$ $m \leq 10$ for s running in the different ranges. One can verify that the content of this Table can be obtained with the help of the formulas (52).

s		S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
1		1	1	1	1	1	1	1	1	1	1
2		1	2	2	2	2	2	2	2	2	2
3		1	2	3	3	3	3	3	3	3	3
4		1	3	4	5	5	5	5	5	5	5
5		1	3	5	6	7	7	7	7	7	7
6		1	4	7	9	10	11	11	11	11	11
7		1	4	8	11	13	14	15	15	15	15
8		1	5	10	15	18	20	21	22	22	22
9		1	5	12	18	23	26	28	29	30	30
10		1	6	14	23	30	35	38	40	41	42
51		1	26	243	1215	4033	9975	19928	33940	51294	70760
52		1	27	252	1285	4319	10829	21873	37638	57358	79725
53		1	27	261	1350	4616	11720	23961	41635	64015	89623
54		1	28	271	1425	4932	12692	26226	46031	71362	100654
55		1	28	280	1495	5260	13702	28652	50774	79403	112804
56		1	29	290	1575	5608	14800	31275	55974	88252	126299
57		1	29	300	1650	5969	15944	34082	61575	97922	141136
58		1	30	310	1735	6351	17180	37108	67696	108527	157564
59		1	30	320	1815	6747	18467	40340	74280	120092	175586
60		1	31	331	1906	7166	19858	43819	81457	132751	195491
101		1	51	901	8262	48006	198230	628998	1621248	3539452	6757864
102		1	52	919	8505	49806	207338	662708	1719877	3778074	7254388
103		1	52	936	8739	51649	216705	697870	1823402	4030512	7782608
104		1	53	954	8991	53550	226479	734609	1932418	4297682	8345084
105		1	53	972	9234	55496	236534	772909	2046761	4580087	8942920
106		1	54	990	9495	57501	247010	812893	2167057	4878678	9578879
107		1	54	1008	9747	59553	257783	854546	2293142	5194025	10254199
108		1	55	1027	10018	61667	269005	898003	2425678	5527168	10971900
109		1	55	1045	10279	63829	280534	943242	2564490	5878693	11733342
110		1	56	1064	10559	66055	292534	990404	2710281	6249733	12541802


1.002
1.001
1
0.999
0.998
1
100
200
300
400
500
Nx10-3